Exercise 1. ***1-loop renormalised QCD coupling constant***

To renormalise QCD one re-scales the fields and parameters in such a way that the Lagrangian can be written: $\mathcal{L}_R + \mathcal{L}_{c.t}$, where $\mathcal{L}_R$ has the form of the original 'bare' Lagrangian $\mathcal{L}_0$, but with each term replaced by it's corresponding renormalised one, and $\mathcal{L}_{c.t}$ contains the respective counter-terms. In particular, under the re-scaling:

$$A_0^{\mu a} = Z_A^{\frac{1}{2}} A_R^{\mu a} \qquad \psi_0 = Z_2^{\frac{1}{2}} \psi_R$$

the bare quark-quark-gluon vertex is transformed:

$$g_0 \bar{\psi}_0 \gamma^\mu T^a \psi_0 A^a_{0\mu} \;\longrightarrow\; g_0 Z_2 Z_A^{\frac{1}{2}} \bar{\psi}_R \gamma^\mu T^a \psi_R A^a_{R\mu}$$

In order to write this in the canonical 'renormalised + counter-term' form one defines the quark-quark-gluon vertex renormalisation factor Z_{1F} and renormalised coupling g_R parameter as follows:

$$g_0 Z_2 Z_A^{\frac{1}{2}} \bar{\psi}_R \gamma^\mu T^a \psi_R A^a_{R\mu} \equiv Z_{1F} g_R \bar{\psi}_R \gamma^\mu T^a \psi_R A^a_{R\mu}$$

(a) By performing dimensional regularisation in $d = 4 - 2\epsilon$ dimensions show that the bare coupling parameter can be written:

$$g_0 = Z_2^{-1} Z_A^{-\frac{1}{2}} Z_{1F} g_R \; \mu^\epsilon$$

where μ is the renormalisation scale.

(b) Using the definition $Z_i = 1 + \delta Z_i$ for $i = A, 2, 1F$, expand the renormalisation factors to $\mathcal{O}(g_R^2)$ and show that:

$$g_0 = \left(1 - \delta Z_2 - \frac{1}{2}\delta Z_A + \delta Z_{1F}\right) g_R \; \mu^\epsilon$$

Then use the relevant results in Exercise sheets 10 and 11 to show that this has the explicit form:

$$g_0 = \left[1 - \frac{g_R^2}{16\pi^2}\left(\frac{11}{6}N_C - \frac{1}{3}n_f\right)\left(\frac{1}{\epsilon} + \text{finite}\right)\right] g_R \; \mu^\epsilon$$

Exercise 2. ***The 1-loop Beta function in QCD***

The Beta function $\beta(\mu)$ in renormalised QCD determines how the coupling parameter $g_R(\mu)$ depends on the renormalisation scale μ via the following differential equation:

$$\frac{\partial g_R}{\partial \ln\mu} = \mu \frac{\partial g_R}{\partial \mu} = \beta(\mu) \tag{1}$$

where the derivatives are taken with g_0 held constant.

(a) With this definition use the result of Exercise 1 part (b) to show that:

$$\beta = -\left(\frac{11}{3}N_C - \frac{2}{3}n_f\right)\frac{g_R^3}{16\pi^2} + \mathcal{O}(g_R^5)$$

(b) By defining a scale Λ_{QCD} via the condition:

$$\lim_{\mu \to \Lambda_{QCD}} \frac{1}{g_R^2(\mu)} = 0$$

solve equation 1 explicitly and show that $\alpha_S(\mu) := \frac{g_R^2}{4\pi}$ satisfies:

$$\alpha_S(\mu) = \frac{4\pi}{\left(\frac{11}{3}N_C - \frac{2}{3}n_f\right)\ln\left(\frac{\mu^2}{\Lambda_{QCD}^2}\right)}$$

(c) Classify all the different possible behaviours of $\alpha_S(\mu)$ as a function of N_C and n_f. What is the form of $\alpha_S(\mu)$ in the case of QCD? Why is this so significant?

Hint. `http://www.nobelprize.org/nobel_prizes/physics/laureates/2004/`, *"The Nobel Prize in Physics 2004". Nobelprize.org. Nobel Media AB 2013. Web. 19 May 2014.*