

**Exercise 11.1 Magnetic monopoles**

We insert the ansatz

$$\phi_a = \frac{H(\xi)}{\xi} \eta \frac{x_a}{r}, \quad A_i^a = \frac{\epsilon_{aij} x_j}{gr^2} (K(\xi) - 1)$$

1. We remark

$$\partial_i \frac{x_a}{r} = \frac{\delta_{ai}}{r} - \frac{x_i x_a}{r^3}, \quad \partial_i \xi = g \eta \frac{x_i}{r} = g^2 \eta^2 \frac{x_i}{\xi}, \quad \partial_i r^n = n x_i r^{n-2}.$$

We calculate the parts in order:

$$\begin{aligned} \partial_i \phi_a &= \partial_i \left( \frac{H}{\xi} \eta \frac{x_a}{r} \right) \\ &= \eta \left( \frac{x_i x_a}{r^3} \xi \left[ \frac{H'}{\xi} - \frac{H}{\xi^2} \right] + \frac{H}{\xi} \left[ \frac{\delta_{ai}}{r} - \frac{x_i x_a}{r^3} \right] \right) \\ &= \eta \left( \frac{x_i x_a}{r^3} \left[ H' - 2 \frac{H}{\xi} \right] + \frac{\delta_{ai}}{r} \frac{H}{\xi} \right) \\ -g \epsilon_{abc} A_i^b \phi_c &= -g \epsilon_{abc} \frac{\epsilon_{bik} x_k}{gr^2} (K - 1) \frac{H}{\xi} \eta \frac{x_c}{r} \\ &= -\eta \frac{H}{\xi} \frac{1}{r^3} (K - 1) (x_i x_a - r^2 \delta_{ai}) \\ &= \eta \left( \frac{x_i x_a}{r^3} \left[ -\frac{H}{\xi} (K - 1) \right] + \frac{\delta_{ai}}{r} \frac{H}{\xi} (K - 1) \right) \end{aligned}$$

adding up to

$$\begin{aligned} \partial_i \phi_a - g \epsilon_{abc} A_i^b \phi_c &= \eta \left( \frac{x_i x_a}{r^3} \left[ H' - \frac{H}{\xi} - \frac{HK}{\xi} \right] + \frac{\delta_{ai}}{r} \frac{H}{\xi} K \right) \\ &= KH \frac{1}{gr^4} (r^2 \delta_{ai} - x_i x_a) + \frac{x_i x_a}{gr^4} (\xi H' - H). \end{aligned}$$

2. For the square we need

$$\begin{aligned} (r^2 \delta_{ai} - x_i x_a)(r^2 \delta_{ai} - x_i x_a) &= r^4 \delta_{ii} + x_i x_i x_a x_a - 2r^2 x_i x_i = 2r^4 \\ (r^3 \delta_{ai} - x_i x_a)x_i x_a &= r^4 - r^4 = 0 \end{aligned}$$

and therefore we have for the kinetic term

$$\frac{1}{2} (D_\mu \phi)(D^\mu \phi) = \frac{-1}{2} (D_i \phi)(D_i \phi) = \frac{-1}{2} \frac{g^2 \eta^4}{\xi^4} (2K^2 H^2 + (\xi H' - H)^2).$$

We proceed with the field strength term, we have

$$\begin{aligned}
\partial_i A_j^a &= \frac{1}{g} \epsilon_{ajk} \partial_i \left( x_k \frac{1}{r^2} (K-1) \right) \\
&= \frac{1}{g} \epsilon_{ajk} \left( \delta_{ik} r^{-2} (K-1) - 2x_i x_k r^{-4} (K-1) + x_k r^{-2} K' g \eta \frac{x_i}{r} \right) \\
&= \frac{1}{g} \epsilon_{aji} \frac{K-1}{r^2} + \frac{1}{g} x_i \epsilon_{ajk} x_k (-2r^{-4} (K-1) + g \eta r^{-3} K') \\
&= \frac{-1}{g} \frac{K-1}{r^2} \epsilon_{aij} + \frac{1}{g} x_i \epsilon_{ajk} x_k r^{-4} (\xi K' - 2(K-1))
\end{aligned}$$

and therefore

$$\partial_i A_j^a - \partial_j A_i^a = \frac{-2}{g} \frac{K-1}{r^2} \epsilon_{aij} + \frac{1}{g} (x_i \epsilon_{ajk} x_k - x_j \epsilon_{aik} x_k) r^{-4} (\xi K' - 2(K-1)).$$

For the non-abelian part we have

$$\begin{aligned}
\epsilon_{abc} A_i^b A_j^c &= \epsilon_{abc} \epsilon_{bik} x_k \epsilon_{cjm} x_m \frac{1}{g^2} \frac{1}{r^4} (K-1)^2 \\
&= (\delta_{ci} \delta_{ak} - \delta_{ck} \delta_{ai}) x_k \epsilon_{cjm} x_m \frac{1}{g^2} \frac{1}{r^4} (K-1)^2 \\
&= x_a \epsilon_{ijm} x_m \frac{1}{g^2} \frac{1}{r^4} (K-1)^2
\end{aligned}$$

giving us

$$F_{ij}^a = \frac{-2}{g} \frac{K-1}{r^2} \epsilon_{aij} + \frac{1}{g} (x_i \epsilon_{ajk} x_k - x_j \epsilon_{aim} x_m) r^{-4} (\xi K' - 2(K-1)) - \frac{1}{g} \frac{(K-1)^2}{r^4} x_a \epsilon_{ijm} x_m.$$

In preparation for the contraction, we calculate

$$\begin{aligned}
\epsilon_{aij} \epsilon_{aij} &= 6 & x_i \epsilon_{ajk} x_k x_i \epsilon_{ajm} x_m &= 2r^4 \\
x_i \epsilon_{ajk} x_k x_j \epsilon_{ajm} x_m &= 0 & (x_i \epsilon_{ajk} x_k - x_j \epsilon_{aik} x_k)^2 &= 4r^4 \\
x_a x_a \epsilon_{ijm} \epsilon_{ijk} x_m x_k &= 2r^4 & \epsilon_{aij} (x_i \epsilon_{ajk} x_k - x_j \epsilon_{aik} x_k) &= -4r^2 \\
\epsilon_{aij} x_a \epsilon_{ijm} x_m &= 2r^2 & (x_i \epsilon_{ajk} x_k - x_j \epsilon_{aik} x_k) x_a \epsilon_{ijm} x_m &= 0
\end{aligned}$$

We insert this into  $F_{ij}^a F^{ij,a} = F_{ij}^a F_{ij}^a = (F_{ij}^a)^2$ :

$$\begin{aligned}
(F_{ij}^a)^2 &= \frac{24}{g^2} \frac{(K-1)^2}{r^4} + \frac{1}{g^2} 4r^{-4} (\xi K' - 2(K-1))^2 + \frac{1}{g^2} \frac{(K-1)^4}{r^8} \cdot 2r^4 \\
&\quad - \frac{4}{g^2} (K-1) (\xi K' - 2(K-1)) (-4r^2) r^{-6} + \frac{4}{g^2} \frac{(K-1)^3}{r^6} \cdot 2r^2 \\
&= \frac{1}{g^2 r^4} [8(K-1)^2 + 2(K-1)^4 + 8(K-1)^3 + 4(\xi K')^2] \\
&= \frac{1}{g^2 r^4} [2(K^2 - 1)^2 + 4(\xi K')^2] \\
&\quad - \frac{1}{4} (F_{ij}^a)^2 = \frac{-g^2 \eta^4}{2\xi^4} [(K^2 - 1)^2 + 2(\xi K')^2]
\end{aligned}$$

We do need the potential term as well, we have

$$\frac{-\lambda}{8} (\phi^2 - \eta^2)^2 = \frac{-\lambda}{8} \frac{\eta^4}{\xi^4} (H^2 - \xi^2)^2.$$

Now we add things up to arrive at the energy

$$\begin{aligned}
E &= - \int d\mathbf{x} \mathcal{L} = - \int \frac{4\pi}{g^3\eta^3} \xi^2 d\xi \mathcal{L} = \frac{4\pi}{g^3\eta^3} \int \xi^2 d\xi \left( \frac{1}{4} F_{\mu\nu}^a F^{\mu\nu,a} - \frac{1}{2} D_\mu \phi D^\mu \phi + \frac{\lambda}{8} (\phi^2 - \eta^2)^2 \right) \\
&= \frac{4\pi\eta}{g} \int d\xi \frac{1}{\xi^2} \left( \frac{1}{2} (K^2 - 1)^2 + (\xi K')^2 + K^2 H^2 + \frac{1}{2} (\xi H' - H)^2 + \frac{\lambda}{8g^2} (H^2 - \xi^2)^2 \right).
\end{aligned}$$

3. We assume  $E[g] = \int f(g(x), g'(x)) dx$  and we recall

$$\begin{aligned}
\frac{\delta E}{\delta g}[\phi] &= \frac{d}{d\epsilon} \int f(g + \epsilon\phi, g' + \epsilon\phi') dx \Big|_{\epsilon=0} \\
&= \int \frac{\partial f(g, g')}{\partial g} \phi + \frac{\partial f(g, g')}{\partial g'} \phi' dx \\
&= \int \left( \frac{\partial f(g, g')}{\partial g} - \frac{\partial}{\partial x} \frac{\partial f(g, g')}{\partial g'} \right) \phi dx
\end{aligned}$$

where we have assumed that the boundary terms of the partial integration vanish. Therefore we get from the condition that the derivatives of the energy with respect to  $K$  and  $H$  should vanish:

$$\begin{aligned}
\frac{\delta E}{\delta K} = 0 &\Rightarrow K'' = \frac{1}{\xi^2} (K(K^2 - 1) + 2KH^2) \\
\frac{\delta E}{\delta H} = 0 &\Rightarrow H'' - \frac{H'}{\xi} + \frac{H}{\xi^2} = \frac{2K^2 H}{\xi^2} - \frac{H'}{\xi} + \frac{H}{\xi^2} + \frac{1}{\xi^2} \frac{\lambda}{2g^2} H(H^2 - \xi^2).
\end{aligned}$$