

Exercise 9.1 Pure gauge

Show that in the case where the gauge field is given by a pure gauge transformation,

$$A_\mu = \frac{i}{g} (\partial_\mu \Lambda) \Lambda^{-1}$$

where Λ is a $SU(N)$ matrix, then the field strength tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu]$ is identically zero.

Exercise 9.2 Instantons

In 4D Euclidean space ($\mu, \nu = 1, 2, 3, 4$), consider the instanton configuration of a $SU(2)$ gauge field

$$A_\mu^a = \frac{2}{g} \frac{\eta_{\mu\nu}^a x_\nu}{|x|^2 + \lambda^2}$$

where λ is an arbitrary distance and η is 't Hooft's mixed colour and space-time tensor defined as

$$\eta_{\mu\nu}^a = \epsilon_{a\mu\nu 4} - \delta_{\mu 4} \delta_{a\nu} + \delta_{\nu 4} \delta_{a\mu}$$

or equivalently

$$\eta_{\mu\nu}^a = \begin{cases} \epsilon_{a\mu\nu} & \text{if } \mu \neq 4, \nu \neq 4 \\ \delta_{a\mu} & \text{if } \mu \neq 4, \nu = 4 \\ -\delta_{a\nu} & \text{if } \mu = 4, \nu \neq 4 \\ 0 & \text{if } \mu = \nu = 4 \end{cases}$$

Some of the properties of η :

$$\begin{aligned} \eta_{\mu\nu}^a &= -\eta_{\nu\mu}^a \\ \epsilon_{abc} \eta_{\mu\nu}^b \eta_{\rho\sigma}^c &= \delta_{\mu\rho} \eta_{\nu\sigma}^a - \delta_{\mu\sigma} \eta_{\nu\rho}^a - \delta_{\nu\rho} \eta_{\mu\sigma}^a + \delta_{\nu\sigma} \eta_{\mu\rho}^a \\ \eta_{\mu\nu}^a \eta_{\mu\nu}^a &= 12 \end{aligned}$$

1. Show that the field strength tensor is given by

$$F_{\mu\nu}^a = \frac{4}{g} \frac{-\lambda^2 \eta_{\mu\nu}^a}{(|x|^2 + \lambda^2)^2}$$

with $F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g\epsilon_{abc} A_\mu^b A_\nu^c$.

2. Compute the action corresponding to this field configuration:

$$\underline{S} = \frac{1}{4} \int d^4x \underline{F}_{\mu\nu}^a \underline{F}_{\mu\nu}^a$$